Analytical Mechanics

P2204

Version Française

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Chapter 1

Rappel des lois de la mecanique Newtonienne

1.1 Enonces des lois de Newton

- loi d'inertie
  Une particule isolee , sur la quelle n'agit aucune force exterieur , rest au repos ou conserve un mouvement recteligne uniform

$$\boxed{\overrightarrow{F} = \overrightarrow{0} \Longleftrightarrow \overrightarrow{v} = \overrightarrow{cte}}$$

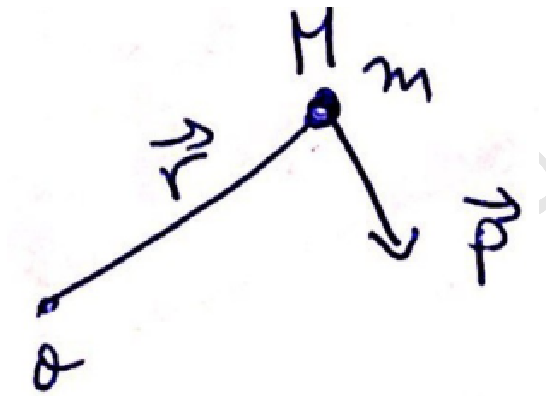
- loi fondamentale de la dynamique
  - $\overrightarrow{F} = m\overrightarrow{a}$
    * $\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$
    * $\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt}$
    * $\overrightarrow{r} = \overrightarrow{OM}$ (O est l origin , M est la point ou F agit)
  - $\overrightarrow{F} = \frac{d\overrightarrow{P}}{dt}$(avec $\overrightarrow{P} = m\overrightarrow{v}$ quantite de mouvement)
- principe de la action et de la reaction $\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$

1.2 Loi de conservaton pour un point materiel

- Conservation de la quantite de mouvement
  si $\vec{F} = 0$, $\vec{F} = \frac{d\vec{P}}{dt} = \vec{0} \implies \vec{P} = \overrightarrow{cte}$

- Conservation du moment angulair

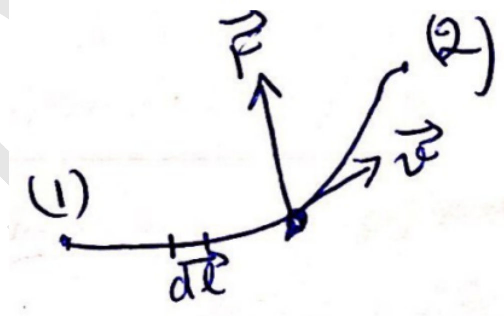
  $\overrightarrow{L_o} = \vec{r} \wedge \vec{p}$
  $\frac{d\overrightarrow{L_o}}{dt} = \vec{r} \wedge \vec{F}$
  si $\vec{F} = \vec{0} \implies L = \overrightarrow{cte}$
  si $\vec{F}$ est porte par $\vec{r} \implies \vec{L} = \overrightarrow{cte}$



- Conservation de lenergie mecanique total

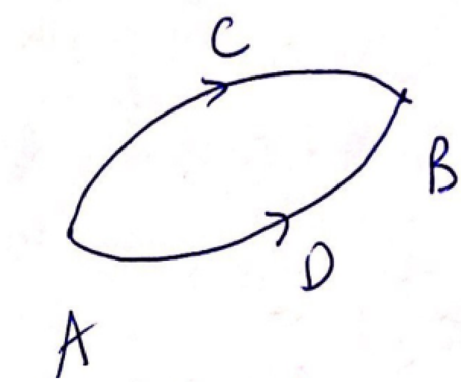
-   - Theorem d energie cinetique

    $\Delta E_c = W$, $W = -\int \vec{F} d\vec{l}$



  - Forces conservatives

    si $W_{ACB} = W_{ADB} \implies$ les forces exterieur sont conservatives

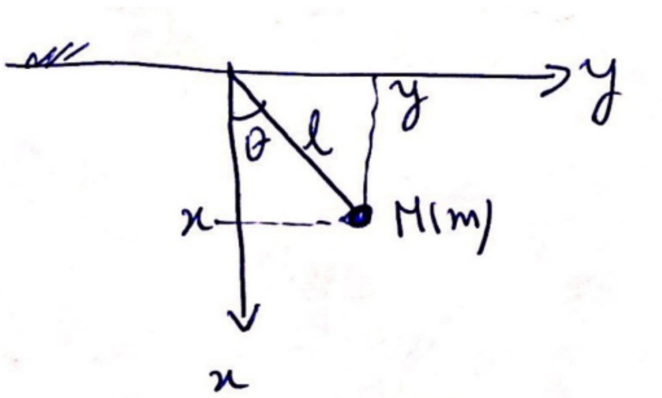


  - Une condition necessaire et suffisant pour que W_{AB} soit independant du chemin est que $\vec{F}$ derive d un potentielle
    $\vec{F} = -\overrightarrow{grad}(U)$ (U : energie potentielle)

1.3 Contraintes et coordonees generalisees

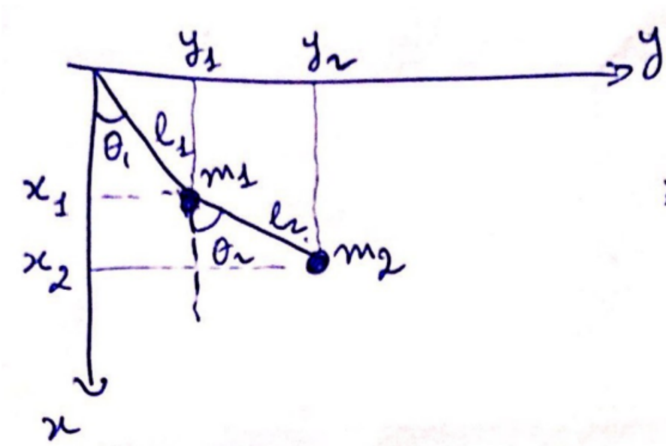
Les contraintes du systeme introduisent des dependances entre les coordonee
les contraintes sont par exemple des hypothese de rigidite , limitation de sont cadre d evolution ...
exemple :

- pendule simple



$$\boxed{x^2 + y^2 = l^2}$$

- pendule simple



$$\boxed{x_1^2 + y_1^2 = l_1^2}$$

$$\boxed{(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2}$$

System de N particule

- Aucun contraint (independante) $\implies$ 3N coordone (3N degree de liberte)
- K contrainte $\implies$ 3N -k coordonees independant

Chapter 2

Lagrange

2.1 Holonomic systeme

A system in which one can deduce the state of a system by knowing only information about the change of positions of the components of the system over time

2.2 Calcule differentielle

Soit f une fonctino de N variables $f = f(r_1 \ldots, r_N)$

- la differentielle totale de f est : $df = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} dr_i$
- la derive de f par rapport a lune de ses variable (r_j) :

$$\frac{df}{dr_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{dr_i}{dr_j}$$

- si tout les variable sont independant

$$\frac{df}{dr_j} = \frac{\partial f}{\partial r_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{dr_i}{dr_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{\partial r_i}{\partial r_j}$$

2.3 Equation generale de lagrange

On considere un system holonome de N particule , d degre du liberte , avex $\overrightarrow{F_\alpha}$ est la force appliquee sur la particule α

2.3.1 lenergie cinetique est :

$T = \sum_{\alpha=1}^{N} T_\alpha = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \overrightarrow{\dot{r_\alpha}}^2$

- $\overrightarrow{r_\alpha} = \overrightarrow{r_\alpha}(q_1, q_2, q_3 \ldots q_d, t)$
- $\alpha = 1, 2, 3 \ldots N$
- q_i: coordone generalise
- $i = 1, 2, 3 \ldots d$
- $\overrightarrow{\dot{r}} = \frac{d\overrightarrow{r_\alpha}}{dt}$

$T_\alpha = T_\alpha(\underbrace{q_1, q_2 \ldots q_d}_{\text{position}}, \underbrace{\dot{q_1}, \dot{q_2} \ldots \dot{q_d}}_{\text{vitess}}, \underbrace{t}_{\text{temp}})$

2.3.2 les forces generalise associe a q_i

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_i}}\right) - \frac{\partial T}{\partial q_i}}$$

Preuve :

- on a :$dT = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} d\dot{r_\alpha}$ (differentielle totale de T) $\implies \frac{dT}{d\dot{q_i}} = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} \frac{d\dot{r_\alpha}}{d\dot{q_i}}$
- Puisque les variable sont independant alors :
  $\frac{dT}{d\dot{q_i}} = \frac{\partial T}{\partial \dot{q_i}} = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} \frac{\partial \dot{r_\alpha}}{\partial \dot{q_i}} \implies \frac{\partial T}{\partial \dot{q_i}} = \frac{\partial}{\partial \dot{q_i}}(\sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha (\overrightarrow{\dot{r_\alpha}})^2) = \frac{1}{2}(\sum_{\alpha=1}^{N} m_\alpha \frac{\partial}{\partial \dot{q_i}} (\overrightarrow{\dot{r_\alpha}})^2)$
  $= \frac{1}{2} \times 2 \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r_\alpha}} \frac{\partial \overrightarrow{\dot{r_\alpha}}}{\partial \dot{q_i}} \implies \boxed{\frac{\partial T}{\partial \dot{q_i}} = \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r_\alpha}} \frac{\partial \overrightarrow{\dot{r_\alpha}}}{\partial \dot{q_i}}}$

- Chercher $\vec{\dot{r}}_\alpha$, ona :
  - $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2 \dots q_d, t)$
  - $\vec{\dot{r}}_\alpha = \frac{d((\vec{r}_\alpha))}{dt}$

  $d\vec{r}_\alpha = \frac{\partial \vec{r}_\alpha}{\partial q_1} dq_1 + \frac{\partial \vec{r}_\alpha}{\partial q_2} dq_2 + \dots + \frac{\partial \vec{r}_\alpha}{\partial q_d} dq_d + \frac{\partial \vec{r}_\alpha}{\partial t} dt$
  $\frac{d\vec{r}_\alpha}{dt} = \vec{\dot{r}}_\alpha = \frac{\partial \vec{r}_\alpha}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_\alpha}{\partial q_2}\frac{dq_2}{dt} + \dots + \frac{\partial \vec{r}_\alpha}{\partial q_d}\frac{dq_d}{dt} + \frac{\partial \vec{r}_\alpha}{\partial t}$
  $\frac{d\vec{r}_\alpha}{dt} = \vec{\dot{r}}_\alpha = \frac{\partial \vec{r}_\alpha}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial \vec{r}_\alpha}{\partial q_2}\frac{\partial q_2}{\partial t} + \dots + \frac{\partial \vec{r}_\alpha}{\partial q_d}\frac{\partial q_d}{\partial t} + \frac{\partial \vec{r}_\alpha}{\partial t}$

$$\boxed{\vec{\dot{r}}_\alpha = \sum_{i=1}^{d} \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_\alpha}{\partial t}} \text{ avec } \alpha = 1, 2 \dots N$$

- demontrer que $\frac{\partial \vec{\dot{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \vec{r}_\alpha}{\partial q_i}$
  $\vec{\dot{r}}_\alpha = \sum_{i=1}^{d} \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial r_\alpha}{\partial t} = \frac{\partial \vec{r}_\alpha}{\partial q_1}\dot{q}_1 + \dots + \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i + \dots + \frac{\partial \vec{r}_\alpha}{\partial q_d}\dot{q}_d + \frac{\partial \vec{r}_\alpha}{\partial t}(q_i, t)$
  $\frac{\partial}{\partial \dot{q}}\left(\frac{\partial \vec{r}}{\partial t}\right) = 0$ car $\frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{r}}{\partial t}(q_i, t)$
  $\frac{\partial \vec{\dot{r}}_\alpha}{\partial \dot{q}_i} = 0 + 0 + 0 \dots + \frac{\partial \vec{r}_\alpha}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{q}_i} + \dots + 0$
  $= \frac{\partial \vec{r}_\alpha}{\partial q_i} = \frac{\partial \vec{r}_\alpha}{\partial q_i} \implies \boxed{\frac{\partial \vec{\dot{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \vec{r}_\alpha}{\partial q_i}}$

- alors $\frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \frac{\partial \vec{r}_\alpha}{\partial \dot{q}_i} \implies \boxed{\frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}}$

- Calcule de $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$
  - $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \sum_{\alpha=1}^{N} m_\alpha \vec{\ddot{r}}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} + \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \underbrace{\frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right)}$
  - Calcule de $\frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right)$
    $d\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) dq_i + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) dt$
    $\frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) \implies \frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial q_i}\left(\frac{\partial \vec{r}_\alpha}{\partial t}\right)$
    $\frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\sum_{i=1}^{d} \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial t}\vec{r}_\alpha\right) = \frac{\partial}{\partial q_i}\vec{\dot{r}}_\alpha = \frac{\partial}{\partial q_i}\frac{d}{dt}(\vec{r}_\alpha) \implies \boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\vec{\dot{r}}_\alpha}$
  - Verifier que $\sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \frac{\partial \vec{\dot{r}}_\alpha}{\partial q_i} = \frac{\partial T}{\partial q_i}$
    on a $T = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha^{\,2} \implies \frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} m_\alpha \frac{\partial}{\partial q_i}(\vec{\dot{r}}_\alpha)^2 \implies \boxed{\frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}}_\alpha \frac{\partial \vec{\dot{r}}_\alpha}{\partial q_i}}$

  alors

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i} + \sum_{\alpha=1}^{N} m_\alpha \vec{\ddot{r}}_\alpha \frac{\partial}{\partial q_i}\vec{r}_\alpha}$$

- on utilise la loi de Newton : $\vec{F}_\alpha = m_\alpha \vec{\ddot{r}}_\alpha \implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i} + \sum_{\alpha=1}^{N} \vec{F}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}$
  avec $\sum_{\alpha=1}^{N} \vec{F}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} = Q_i$

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}}$$

2.4 Formalisme de lagrange : cas d un system conservatif

- Si les forces $\overrightarrow{F_\alpha}$ sont conservatif , On suppose que toutes les forces agissant sur ce system derivent d'ume mem energie potentielle U
  avex U depend de position $U = U(\overrightarrow{r_1}, \overrightarrow{r_2} \ldots, \overrightarrow{r_N})$ et $\overrightarrow{F_\alpha} = -\overrightarrow{grad}U$
  Donc , on a $\boxed{\sum_{\alpha=1}^{N} \overrightarrow{F_\alpha} d\overrightarrow{r_\alpha} = -dU}$
- On a l equation de lagrange $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} \overrightarrow{F_\alpha} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} = Q_i$
  avec $Q_i = \sum_{\alpha=1}^{N} \overrightarrow{F_\alpha} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} = \frac{-dU}{dq_i} \implies \boxed{Q_i = \frac{-\partial U}{\partial q_i}}$(si les forces sont conservatives)
- L'equation devien $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{-\partial U}{\partial q_i} \implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{-\partial}{\partial q_i}(T - U) = 0$
- U depend seulment du position alors $\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}(T - U)$
  l equation devien $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}(T - U)\right) - \frac{\partial}{\partial q_i}(T - U) = 0$
- On introdui la fonction de lagrange (lagrangien)

$$\boxed{L(qi, \dot{q}_i, t) = T - U}$$

T depend de vitess $\dot{q}_i$ et U depend de position q_i

- L equation devien

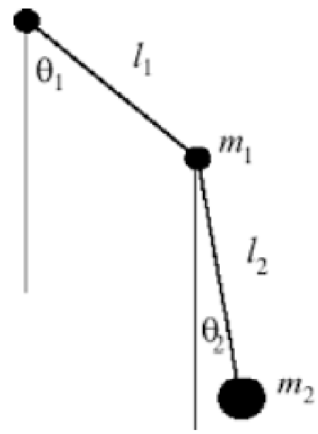
Equation Euler-lagrange : $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$

2.5 Les lois de conservation

- Invariance par temps : ENERGIE
  pour le demontre :$\frac{\partial L}{\partial t} = 0 \implies$ t est un variable cyclique
  on applique belltrami : $L - \frac{\dot{q}\partial L}{\partial \dot{q}} = cte$ ... jusqua arrive a $U + T = cte$
- Invariance par translation : quantite de mouvement (translation)
  pour le demontre : si q_α est un variable cyclique $\implies \frac{\partial L}{\partial \alpha} = 0 \implies$ la quantite du mouvement $p_\alpha = cte$
- Invariance par rotation : MOMENT CINETIQUE (rotation)
  pour le demontre : Si θ une coordone generalise angle est cyclique $\implies \vec{\mathfrak{L}} = cte$
  note que le moment cinetique est : $\vec{\mathfrak{L}} = \vec{r} \wedge \vec{p}$

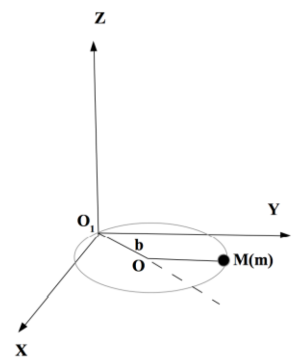
2.6 The key in somme problems

2.6.1 Pendule double



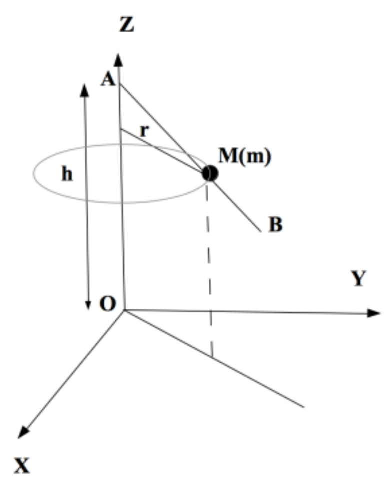
- $\overrightarrow{OM_1} = l_1\overrightarrow{u_{r_1}}$
  $\implies v_1 = \frac{d}{dt}l_1\overrightarrow{u_{r_1}} = l_1\dot{\theta}\overrightarrow{u_\theta}$
  $\implies v_1^2 = l_1^2\dot{\theta}_1^{\ 2}$
- $\overrightarrow{OM_2} = OM_1 + M_1M_2 = l_1\overrightarrow{u_{r_1}} + l_2\overrightarrow{u_{r_2}}$
  $\implies v_2 = l_1\dot{\theta}_1\overrightarrow{u_{\theta_1}} + l_2\dot{\theta}_2\overrightarrow{u_{\theta_2}}$
  $\implies v_2^2 = (l_1\dot{\theta}_1)^2 + 2(l_1\dot{\theta}_1)(l_2\dot{\theta}_2)\overrightarrow{u_{\theta_1}}\overrightarrow{u_{\theta_2}} + (l_2\dot{\theta}_2)^2$
  $= (l_1\theta_1)^2 + 2(l_1\dot{\theta}_1)(l_2\dot{\theta}_2)\cos(\theta_2 - \theta_1) + (l_2\dot{\theta}_2)^2$

2.6.2 eh ...



$O_1M = O_1O + OM$
$v = bw\overrightarrow{u_\varphi} + b(w + \dot{\theta})\overrightarrow{u_\theta}$
$v^2 = b^2(w^2 + 2w\overrightarrow{u_\varphi}(w + \dot{\theta})\overrightarrow{u_\theta} + (w + \dot{\theta})^2)$
$= b^2(w^2 + 2w(w + \dot{\theta})\cos(\theta) + (w + \dot{\theta})^2)$

2.6.3 mmm ...



$v = \dot{r}\overrightarrow{u_r} + (r\sin(\alpha)w.\overrightarrow{u_\theta})$
$v^2 = \dot{r}^2 + 2\dot{r}\sin(\alpha)w\cos(\frac{\pi}{2}) + r^2w^2\sin(\alpha)^2$
$= \dot{r}^2 + r^2w^2\sin(\alpha)^2$

2.7 Beltrami & Moindre d'action

2.7.1 Moindre d'action

$\delta S = \int \delta L dt = 0 \iff \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$

2.7.2 Beltrami

Si t est une variable cyclique alors :
$L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = cte \iff \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$

2.7.3 Lagrange a partir du moindre action

- Soit S est l 'action du systeme
  $S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$
- Le principe de moindre d'action stipule que le systeme evolue de la position q_1 a la position q_2 de telle facon que S est la plus petite valeur possible $\implies \delta S = 0$
  $\delta S = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left[L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t) \right] dt = 0$
- En se limitant dans le developpement de taylor de l'integrant suivant les puissance de δq et $\delta \dot{q}$ aux termes de premier ordre , on obtient :
  $\delta S = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}) dt = 0$
- On a : $\delta \dot{q} = \frac{d}{dt} \delta q$
  $\implies \delta S = \frac{\partial L}{\partial \dot{q}} \delta q]_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}) \delta q dt = 0$
- $\frac{\partial L}{\partial \dot{q}} \delta q]_{t_1}^{t_2} = 0$
- L'integral doit etre nulle pour tout δq , ceci n'est possible que si l'integrant est identiquement null

$$\implies \boxed{\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0}$$

2.7.4 Lagrange a partir de Beltrami

Identite de beltrami : $L - \dot{q} \frac{\partial L}{\partial \dot{q}} = cte$

$\left(L - \dot{q} \frac{\partial L}{\partial \dot{q}} = cte \right) \times \frac{d}{dt}$ avec $\begin{cases} dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} \\ \frac{dL}{dt} = \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} \end{cases}$

$\frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{\partial L}{\partial \dot{q}} \ddot{q} - \dot{q} \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = 0$

$\dot{q} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) \right] = 0$

$\implies \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = 0}$

2.7.5 Trajectoir du lumier

$T = \int dt = \int \frac{dl}{v} = \int \frac{\sqrt{1 + y'^2}}{c} \times n dx$ avec n : indice de refraction $n(x, y)$
(On peut pas utilise beltrami car x n'est pas un variable cyclique (n depend de x))

$\delta T = 0 \implies \int \delta \left(\frac{\sqrt{1+y'^2}}{c} \times n \right) dx = 0$
$f(x, y, y') = \frac{\sqrt{1+y'^2}}{c} \times n \implies \int \delta f dx = 0$
$\implies \frac{d}{dx}(\frac{\partial f}{\partial y'}) - \frac{\partial f}{\partial y} = 0$ (Euler-Lagrange)

- $\frac{\partial f}{\partial y'} = \frac{1}{c} n \frac{1}{2} \frac{2y'}{(1+y'^2)^{\frac{1}{2}}} = \frac{ny'}{c(1+y'^2)^{\frac{1}{2}}}$
- $\frac{d}{dx}(\frac{\partial f}{\partial y'}) = \frac{d}{dx} n(\frac{y'}{c(1+y'^2)^{\frac{1}{2}}}) + \frac{n}{c}\frac{d}{dx}(\frac{y'}{(1+y^2)^{\frac{1}{2}}})$
  $= \left(\frac{\partial n}{\partial y}\frac{dy}{dx} + \frac{\partial n}{\partial x}\frac{dx}{dx}\right)(\frac{y'}{c(1+y'^2)^{\frac{1}{2}}}) + \frac{n}{c}\left(\frac{y''}{\sqrt{1+y'^2}} + y'(\frac{-1}{2})2y'y''(1+y'^2)^{\frac{-3}{2}}\right)$
  $= \left(\frac{\partial n}{\partial y} y' + \frac{\partial n}{\partial x}\right)(\frac{y'}{c\sqrt{1+y'^2}}) + \frac{ny''}{c\sqrt{1+y'^2}} - \frac{n}{c}\frac{y'^2 y''}{(1+y'^2)^{\frac{3}{2}}}$

$\implies \left(\frac{d}{dx}(\frac{\partial f}{\partial y'}) - \frac{\partial f}{\partial y} = 0\right) \times (\sqrt{1+y'^2} \times c)$

$$\implies \boxed{ny'' - \frac{\partial n}{\partial y} + y'\frac{\partial n}{\partial x} - \frac{ny'^2 y''}{\sqrt{1+y'^2}} = 0}$$

Pour n = cte $\implies ny'' - \frac{ny'^2 y''}{(1+y'^2)} = 0 \implies y' = a \implies y = ax + b$

2.7.6 minimize la distance entre deux points

$T = \int dl \implies T = \int \sqrt{1+y'^2} dx \ (dl^2 = dx^2 + dy^2\ )$
$\delta T = \int \delta(\sqrt{1+y'^2}) dx = 0$, Soit $f(y') = \sqrt{1+y'^2} \implies \int \delta f dx = 0$
$\implies f - y'\frac{df}{\partial y'} = cte$ (Beltrami)
$\sqrt{1+y'^2} - y'\frac{\partial\sqrt{1+y'^2}}{\partial y'} = cte$
$\sqrt{1+y'^2} - 2y'^2\frac{1}{2}\frac{1}{\sqrt{1+y'^2}} = cte$
$\frac{1}{\sqrt{1+y'^2}} = cte \implies y' = cte \implies \boxed{y = ax + b}$

2.7.7 minimize le temp entre deux point avec un potentiel

$T = \int dt = \int \frac{dl}{v} = \int \frac{\sqrt{1+y'^2}}{v} dx$
Dapres la conservation de l'energie : $\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv^2 + mgy_m \implies 0 = mgy_m + \frac{1}{2}mv^2$ avec $y_m = -y$
$\implies \boxed{v = \sqrt{2gy}}$
$T = \int \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx$
$\delta T = \int \delta \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx = 0$
$\implies \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y'\frac{\partial \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}}{\partial y'} = cte$ (Beltrami)
$\implies \frac{1}{\sqrt{2gy}}(\frac{1}{\sqrt{1+y'^2}}) = cte$

$$\implies \boxed{1 + y'^2 = \frac{cte}{y}}$$

2.8 Notes

- $\frac{d}{dt}(cos(\theta)) = -\dot{\theta}\sin(\theta)$
- Le moment conjuge : $p_\alpha = \frac{\partial L}{\partial \dot{q_\alpha}}$

Chapter 3

Hamilton

$H = \sum_i p_i \dot{q}_i - L$

Dans le cas d'un system conservatif : $H = T + U$

Considerons un systeme mecanique isole ou conservatif a d degres de liberte , on peut describe se system par :

- Formulation de lagrange :
  d variables independantes $\implies$ d equations de 2eme ordre independantes
- Formuation de Hamilton :
  $2d$ variables independantes $\implies$ $2d$ equations de 1er ordre independantes

3.1 Les equation canoniques de Hamilton

On peut obtenir les equation canoniques de hamilton par la transformation de legendre

- $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$
- $\dot{p}_i = \dfrac{-\partial H}{\partial q_i}$
- $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}$

3.2 Solve a problem

- Trouver U et T
- Trouver H
  - $H = T + U$ (Systeme conservatif)
  - $H = \sum_i p_i \dot{q}_i - L$
- On express H en fonction de p_i et q_i seulment , on replace les valeurs $\dot{q}_i$ en utilison $p_i = \frac{\partial L}{\partial \dot{q}_i}$
- On Cherche $\dot{q}_i$ en utilisant $\dot{q}_i = \frac{\partial H}{\partial p_i}$
- On dervice $\dot{q}_i$ par dt , on obtient $\ddot{q}_i$ en fonction de $\dot{p}_i$

- On Cherche $\dot{p}_i$ en utilisant $\dot{p}_i = \frac{-\partial H}{\partial q_i}$
- $\frac{-\partial H}{\partial q_i}$ en plupart de temp est $\frac{-\partial U}{\partial q_i}$ alors express $\dot{p}_i$ en fonctions des Forces en utilisant $\overrightarrow{F} = -\overrightarrow{grad}(U)$
- remplace les valeur des $\dot{p}_i$ dans les equation des $\ddot{q}_i$

3.3 Crochet de poisson

$\{A, B\}_{q,p} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$
Propriété :

- $\{A, B\}_{pq} = -\{B, A\}_{pq}$
- $\{A + B, C\} = \{A, C\} + \{B, C\}$
- $\frac{\partial}{\partial t}\{A, B\} = \{\frac{\partial A}{\partial t}, B\} + \{A, \frac{\partial B}{\partial t}\}$
- $\{A, cte\} = 0$
- $\{A, A\} = 0$

3.4 Transformation Canonique

En mécanique hamiltonienne, une transformation canonique est un changement des coordonnées canoniques (q, p, t) → (Q, P, t) qui conserve la forme des équations de Hamilton, sans pour autant nécessairement conserver le Hamiltonien en lui-même.

$\boxed{\{Q, P\}_{q,p} = 1 \iff \text{Equation Canonique}}$

3.4.1 Fonction Generatrice

En physique, et plus particulièrement en mécanique hamiltonienne, une fonction génératrice est, de façon lâche, une fonction dont les dérivés partiels génèrent les équations différentielles qui déterminent la dynamique d'un système.

- Fonction generatrice : $F_1(q_i, Q_i)$
  ses dérivées : $p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$
- Fonction generatrice : $F_2(q_i, P_i)$
  ses dérivées : $p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$
- Fonction generatrice : $F_3(p_i, Q_i)$
  ses dérivées : $q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$
- Fonction generatrice : $F_4(p_i, P_i)$
  ses dérivées : $q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$

Toujour : $H' = H + \frac{\partial F}{\partial t}$